

GLOBAL POINCARÉ INEQUALITY ON GRAPHS VIA CONICAL CURVATURE-DIMENSION CONDITIONS

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ABSTRACT. We introduce and study the *conical curvature-dimension condition*, $CCD(K, N)$, for graphs. We show that $CCD(K, N)$ provides necessary and sufficient conditions for the underlying graph to satisfy a sharp global Poincaré inequality which in turn translates to a sharp lower bound for the first eigenvalues of these graphs. An immediate application of the *conical curvature-dimension* analysis is finding a sharp estimate on the curvature of complete graphs. We will then introduce *generalized harmonic maps* and show that when the underlying graph is k -regular, for k large enough compared to the size of G , any function realizing the maximum possible curvature is generalized harmonic. In passing we relate the dimensionless curvature bound of G to that of the cone over G .

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1. INTRODUCTION

The relation between Ricci curvature bounds and the analytic and geometric properties of a smooth Riemannian manifold is a well studied subject in geometric analysis. Thanks to the seminal work

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of Sturm [14] [15] and Lott-Villani [13], the notion of lower Ricci curvature bounds can be generalized to the setting of metric and measure spaces.

A Polish metric measure space that satisfies the Lott-Sturm-Villani's $CD(K, N)$ curvature-dimension conditions is called a $CD(K, N)$ space. One important aspect of these spaces is that they support both local and global Poincaré inequalities (for a sharp global Poincaré inequality and spectral gap on $CD(K, N)$ metric measure spaces, see [12, Theorem 5.34]). For metric measure spaces that satisfy certain infinitesimal regularity properties, the $CD(K, N)$ curvature-dimension bounds coincide with the Bakry-Émery curvature-dimension bounds (or $BE(K, N)$ for short), see [7]. Also, there is a close relation between the lower Ricci bound of X and the lower Ricci curvature bound(s) of the cone(s) over X , when X is a Riemannian manifold with $\text{Ric} \geq (n-1)K$ or more generally an $RCD(K, N)$ space. In particular a Riemannian manifold, X , satisfies $\text{Ric} \geq 1$ if and only if the Riemannian cone over X satisfies $\text{Ric} \geq 0$. In the setting of $RCD(K, N)$ metric measure spaces the relation between the weak Ricci curvature bound of X and that of the cone(s) over X has been explored in [8].

There are some disparities between the discrete Laplacian on graphs and the Laplacian on manifolds (or on some more general non-smooth continuous metric measure spaces). Despite these disparities, studying Bakry-Émery type curvature-dimension conditions for the discrete Laplacian has proven fruitful in the sense that in the discrete setting graphs with lower Ricci curvature bounds satisfy some properties that are similar to the ones satisfied by manifolds with lower Ricci curvature bounds, see [11], [10], [4] and [9].

In this paper we acquire partial results relating the curvature of a graph to the curvature of the cone over it. However, in general our paper does not admit a clean cut relation between the lower Bakry-Émery Ricci curvature bound of the base graph and that of the cone over the graph. This is mainly due to the fact that in the discrete setting the distance between any two vertices in a cone is at most two and the operator Γ_2 at any point x (a key ingredient in the definition of curvature-dimension bounds) will depend on the entire graph. So the curvature bound at the cone point over the vertex set of a graph will store the curvature information of the entire graph, see 1.1.

This article is primarily concerned with the properties of the underlying graph G that can be extracted when the cone over G satisfies the $CD(K, N)$ curvature-dimension conditions at the cone point (a property which will be called the conical curvature-dimension, or $CCD(K, N)$ condition). Our main results are a global Poincaré inequality and the spectral gap estimates that follow.

Definition 1.1 ($CCD(K, N)$ Curvature-Dimension Conditions). Let $G = (V, E)$ be a finite, connected, undirected, loop-edge free graph and consider the cone over the vertex set of G . G is said to satisfy the *conical curvature-dimension condition*, $CCD(K, N)$ for $K \in \mathbb{R}$ and $N \in (1, \infty]$, if the cone over G satisfies the $CD(K, N)$ curvature-dimension conditions at the vertex p , namely if

$$\Gamma_2^c(f)(p) \geq \frac{(\Delta^c f)^2(p)}{N} + K\Gamma_1^c(f)(p), \quad (1)$$

holds for any function f defined on the cone and Δ^c , Γ_1^c and Γ_2^c are the usual Δ , Γ_1 and Γ_2 operators (see (3), (4) and (5)) except on the cone $C(G)$ over G . We note that the second term in (1) is understood to be zero when $N = \infty$.

Now we can state our main theorems and corollaries:

Theorem 1.1 ($CCD(K, N)$ implies global Poincaré Inequality). *If a graph, G , satisfies $CCD(K, N)$, then for any function f on G one has*

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K + |V| - 3}{4} \sum_{y \in V} f^2(y). \quad (2)$$

For functions f with $\text{avg}(f) = 0$, this reduces to the following global Poincaré inequality,

$$\|f\|_2 \leq \sqrt{\frac{2}{2K + |V| - 3}} \|\nabla f\|_2,$$

where $\|\nabla f\|_2$ is understood in the graph setting to be $2 \cdot \sum_{y \in V} \Gamma_1(f)(y)$.

Corollary 1.2. *If G satisfies $CCD(K, N)$ condition, then*

$$\lambda_1(G) \geq \sqrt{\frac{2}{2K + |V| - 3}}.$$

Theorem 1.3. *For any graph, G , and a given $N > 1$, the conical curvature cannot exceed the following number:*

$$K_{max}^c = \frac{|V|}{2} + \frac{3}{2} - 2\frac{|V|}{N}.$$

Corollary 1.2 (Ricci Curvature of Complete Graphs). *Suppose G is the complete graph on n vertices, then the $CD(K, N)$ property coincides with the $CCD(K, N)$ condition on the complete subgraph with $n - 1$ vertices and the curvature of G is $\frac{n}{2} + \frac{1}{2} - 2\frac{(n-1)}{N}$. Furthermore any function that realizes this curvature bound is constant (harmonic).*

Remark 1.4. *When $N = \infty$, our bound $K_{max}^c = 1 + \frac{n}{2}$ coincides with the maximum Ricci curvature of complete graphs as found in [9].*

One can also characterize $CCD(K, N)$ curvature maximizers for general graphs.

Theorem 1.5 (Curvature Maximizers). *Suppose G satisfies $CCD(K_{max}^c, N)$. Then a non-constant function, f , realizes K_{max}^c if and only if $f - \text{avg}(f)$ is an eigenfunction corresponding to $\lambda_1(G) = \frac{N-2}{4N}|V|$. Furthermore if G is k -regular and $k > \frac{N-2}{8N}|V|$, then f is generalized harmonic of types I and II (see Definition 2.5).*

The following theorem illustrates an applications of our Γ -calculus on cones:

Theorem 1.6. *Suppose G satisfies $CD(K, \infty)$ for $K \leq \frac{1}{2}$ then $C(G)$ satisfies $CD(K + 1, \infty)$.*

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2. PRELIMINARIES

Let $G = (V, E)$ be an undirected, unweighted, connected, locally finite graph without any loop-edges. Let $f : V \rightarrow \mathbb{R}$ and consider the space of square-summable functions on the vertex set.

For us the graph Laplacian is given by

$$\Delta f(x) = \sum_{y \sim x} (f(y) - f(x)), \quad (3)$$

where $y \sim x$ means that $(y, x) \in E$. Also, note that the graph Laplacian is a real valued, self-adjoint linear operator (for a thorough treatment of the graph Laplacian see [6]).

Let $F \subset V$, then the boundary of F is

$$\partial F := \{(x, y) \in E \mid x \in F \text{ and } y \in V \setminus F\}.$$

The isoperimetric constant (or Cheeger's constant) is then defined as

$$h(G) := \inf \left\{ \frac{|\partial F|}{\min\{|F|, |V \setminus F|\}} : 0 < |F| < \infty \right\}.$$

A well known generalization of Cheeger's and Buser's results for Riemannian manifolds is the following theorem due to Dodziuk [6] and Alon-Milman [3].

Theorem 2.1 ([6], [3]). *Let $G = (V, E)$ be a finite, connected, edge-loop free graph. Let $d_{\max} = \sup_{v \in V} \{\deg(v)\}$ and let μ_1 be the first non-trivial eigenvalue of the adjacency operator A_G , then*

$$\frac{d_{\max} - \mu_1}{2} \leq h(G) \leq \sqrt{2d_{\max}(d_{\max} - \mu_1)}.$$

The Γ operators of Bakry-Émery associated to the graph Laplacian, Δ , are:

$$\Gamma_1(f, g)(x) = \frac{1}{2} \left[\Delta(fg)(x) - g(x)\Delta f(x) - f(x)\Delta g(x) \right], \quad (4)$$

$$\Gamma_2(f, g)(x) = \frac{1}{2} \left[\Delta \Gamma_1(f, g)(x) - \Gamma_1(\Delta f, g)(x) - \Gamma_1(f, \Delta g)(x) \right]. \quad (5)$$

It is straightforward to check that

$$\Gamma_1(f, g)(x) = \frac{1}{2} \sum_{y \sim x} (f(y) - f(x))(g(y) - g(x)) =: \frac{1}{2} \langle \nabla f, \nabla g \rangle. \quad (6)$$

Throughout these notes $\Gamma_1(f) := \Gamma_1(f, f)$ and similarly $\Gamma_2(f) := \Gamma_2(f, f)$. Thus $\Gamma_1(f)(x) = \frac{1}{2} |\nabla f(x)|^2$ and one can verify the following useful divergence-type identity:

$$\frac{1}{2} \|\nabla f\|_2^2 = \sum_{y \in V} \Gamma_1(f)(y) = - \sum_{y \in V} f(y) \Delta f(y). \quad (7)$$

Definition 2.2 (Bakry-Émery Curvature-Dimension Condition). *Suppose $K \in \mathbb{R}$ and $N \in (1, \infty]$. We say that a graph $G = (V, E)$ satisfies the curvature-dimension conditions, $CD(K, N)$, if for every $x \in V$ and every $f \in \ell^2(V)$,*

$$\Gamma_2(f)(x) \geq \frac{(\Delta f)^2(x)}{N} + K\Gamma_1(f)(x). \quad (8)$$

Note when $N = \infty$, the second term in the inequality above is understood to be 0.

Definition 2.1 (Uniform and Pointwise Ricci Curvatures). We define the dimensional (respectively, dimensionless) Ricci curvature of the graph G , $\text{Ric}_N(G)$ (respectively, $\text{Ric}_\infty(G)$), by

$$\text{Ric}_N(G) := \sup \{K : G \text{ satisfies } CD(K, N)\}$$

and

$$\text{Ric}_\infty(G) := \sup \{K : G \text{ satisfies } CD(K, \infty)\}.$$

Similarly, we define the pointwise curvatures by

$$\text{Ric}_N(y) := \sup \{K : \Gamma_2(f)(y) \geq \frac{1}{N} (\Delta f)^2(y) + K\Gamma_1(f)(y), \forall f\}$$

and

$$\text{Ric}_\infty(y) := \sup \{K : \Gamma_2(f)(y) \geq K\Gamma_1(f)(y), \forall f\}.$$

Definition 2.3 (Conical Ricci Curvatures). *We define the conical Ricci curvature by*

$$CRic_N(G) := \sup\{K : G \text{ satisfies } CCD(K, N) \text{ as in (1)}\}$$

and

$$CRic_\infty(G) := \sup\{K : G \text{ satisfies } CCD(K, \infty) \text{ as in (1)}\}.$$

In section 5 we show that there is a class of functions for which the optimal lower Ricci curvature bound (as defined above) is realized and that these maximizers are generalized harmonic functions (see Definition 2.5 below).

Definition 2.4 (n -th Iterated Means). *Let $f \in \ell^2(V)$ for some graph $G = (V, E)$ and let*

$$g_1(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y), \quad \text{and} \quad h_1(x) = \frac{1}{\deg(x) + 1} \left[f(x) + \sum_{y \sim x} f(y) \right],$$

then for $n > 1$, we define the n -th iterated unit spherical mean and the n -th iterated unit ball mean to be

$$g_n(x) = \frac{1}{\deg(x)} \sum_{y \sim x} g_{n-1}(y), \quad \text{and} \quad h_n(x) = \frac{1}{\deg(x) + 1} \left[h_{n-1}(x) + \sum_{y \sim x} h_{n-1}(y) \right].$$

Definition 2.5 (Generalized Harmonic Functions). *Let G be a finite graph. A function $f : G \rightarrow \mathbb{R}$ is said to be generalized harmonic of type I or II (resp.) if for g_n or h_n (resp.) as in 2.4,*

$$g_n \rightarrow \text{avg}(f) \text{ as } n \rightarrow \infty, \quad \text{or} \quad h_n \rightarrow \text{avg}(f) \text{ as } n \rightarrow \infty. \quad (\text{resp.})$$

We close this section by recalling that the first non-zero eigenvalue of the Laplacian may be computed via the Rayleigh quotient:

$$\lambda_1 = \inf \left\{ \frac{\|\nabla f\|^2}{\|f\|^2} : \text{avg}(f) = 0 \right\}. \quad (9)$$

3. CONES OVER GRAPHS AND THEIR Γ - CALCULUS

The complete cone, $C(G)$, over a finite graph G is constructed by taking the graph Cartesian product of G and H , $G \square H$, where $H = (\{q, p\}, \{(q, p)\})$ is the complete graph on two vertices q and p , and then identifying all the vertices whose second component is p . In this paper p refers to the cone point of $C(G)$.

More generally for a subset, $X \subset V(G)$, the partial cone, $C(X, G)$, is a subgraph of $C(G)$ containing G and all edges (x, p) , $x \in X$. For brevity we will use a superscript c to denote any operation that is taking place in a partial cone over G . Notice that any vertex $v \in V(G)$ can be thought of as the cone point over the 1-sphere based at v , i.e. $S_v^1 := \{y \in V \mid d_G(y, v) = 1\} = X$ in the above construction. In this way partial cones can be useful in studying cliques.

The first subsection is devoted to proving a few lemmas that calculate the Δ and Γ operators of a partial cone in terms of the similar operators on the base graph. The last subsection is devoted to an immediate result.

3.1. Γ -Calculus on a Cone. Since Δ and Γ operators agree for functions that differ by a constant we may assume, without loss of generality, that $f(p) = 0$. Denote by S_p^n and B_p^n the metric spheres and balls (resp.) with radius n and center p in the cone. For any subset $B \subset V$, the notation $v \in B \sim x$ means $v \in B$ and $v \sim x$.

Remark 3.1. *Note that Δ and Γ_1 only depend on vertices that are at most one away. Thus, $\Delta^c f(x) = \Delta f(x)$ and $\Gamma_1^c(f)(x) = \Gamma_1(f)(x)$ when $x \approx p$.*

Lemma 3.2. *Let f be a function on the cone with $f(p) = 0$ then,*

$$\Delta^c f(x) = \begin{cases} \Delta f(x) - f(x); & x \sim p \\ \sum_{y \in S_p^1} f(y); & x = p \end{cases}$$

Proof. (1) If $x \sim p$, then

$$\Delta^c f(x) = \sum_{y \in C \sim x} (f(y) - f(x)) = \sum_{y \in V \sim x} (f(y) - f(x)) + (f(p) - f(x)) = \Delta f(x) - f(x).$$

(2) If $x = p$, then

$$\Delta^c f(p) = \sum_{y \in C \sim p} (f(y) - f(p)) = \sum_{y \in S_p^1} f(y).$$

■

Lemma 3.3. *Let f be a function on the cone with $f(p) = 0$ then,*

$$\Gamma_1^c(f)(x) = \begin{cases} \Gamma_1(f)(x) + \frac{1}{2}f^2(x); & x \sim p \\ \frac{1}{2} \sum_{y \in S_p^1} f^2(y); & x = p \end{cases}$$

Proof. (1) If $x \sim p$, then using (6)

$$\Gamma_1^c(f)(x) = \frac{1}{2} \sum_{y \in C \sim x} (f(y) - f(x))^2 = \frac{1}{2} \sum_{y \in V \sim x} (f(y) - f(x))^2 + \frac{1}{2} (f(p) - f(x))^2 = \Gamma_1(f)(x) + \frac{1}{2} f^2(x).$$

(2) If $x = p$, then using (6)

$$\Gamma_1^c(f)(p) = \frac{1}{2} \sum_{y \in V} (f(y) - f(p))^2 = \frac{1}{2} \sum_{y \in S_p^1} f^2(y).$$

■

In the next few lemmas we calculate the constituent parts that appear in the definition of Γ_2^c .

Remark 3.4. *Note that Γ_2^c depends on vertices at most two away. Thus Γ_2^c coincides with Γ_2 when $x \in V \setminus B_p^2$.*

Lemma 3.5. *Let f be a function defined on the cone, and suppose $f(p) = 0$, then*

$$\Gamma_1^c(f, \Delta^c f)(x) = \begin{cases} \Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_p^1 \sim x} f(y)(f(y) - f(x)); & x \in S_p^2 \\ \Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 \\ \quad + \frac{1}{2} f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) \\ \quad - \frac{1}{2} f(x) \sum_{y \in S_p^1} f(y) + \frac{1}{2} f(x) \Delta f(x) - \frac{1}{2} f^2(x) & x \sim p \\ \frac{1}{2} \sum_{y \in S_p^1} f(y) \Delta f(y) - \frac{1}{2} \sum_{y \in S_p^1} f^2(y) - \frac{1}{2} (\sum_{y \in S_p^1} f(y))^2 & x = p \end{cases}$$

Proof. (1) If $x \in S_p^2$, then using (6)

$$\begin{aligned}
 \Gamma_1^c(f, \Delta^c f)(x) &= \frac{1}{2} \sum_{y \in C \sim x} (f(y) - f(x))(\Delta^c f(y) - \Delta^c f(x)) \\
 &= \frac{1}{2} \sum_{y \in V \setminus S_p^1 \sim x} (f(y) - f(x))(\Delta^c f(y) - \Delta^c f(x)) \\
 &\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))(\Delta^c f(y) - \Delta^c f(x)) \\
 &= \frac{1}{2} \sum_{y \in V \setminus S_p^1 \sim x} (f(y) - f(x))(\Delta f(y) - \Delta f(x)) \\
 &\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))(\Delta f(y) - f(y) - \Delta f(x)) \\
 &= \frac{1}{2} \sum_{y \in V \sim x} (f(y) - f(x))(\Delta f(y) - \Delta f(x)) - \frac{1}{2} \sum_{y \in S_p^1 \sim x} f(y)(f(y) - f(x)) \\
 &= \Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_p^1 \sim x} f(y)(f(y) - f(x)).
 \end{aligned}$$

(2) If $x \sim p$, then using (6)

$$\begin{aligned}
 \Gamma_1^c(f, \Delta^c f)(x) &= \frac{1}{2} \sum_{y \in C \sim x} (f(y) - f(x))(\Delta^c f(y) - \Delta^c f(x)) \\
 &= \frac{1}{2} \sum_{y \in S_p^2 \sim x} (f(y) - f(x))(\Delta f(y) - \Delta f(x) + f(x)) \\
 &\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))(\Delta f(y) - f(y) - \Delta f(x) + f(x)) \\
 &\quad + \frac{1}{2}(f(p) - f(x))\left(\sum_{y \in S_p^1} f(y) - \Delta f(x) + f(x)\right) \\
 &= \frac{1}{2} \sum_{y \in S_p^2 \sim x} (f(y) - f(x))(\Delta f(y) - \Delta f(x)) + \frac{1}{2}f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) \\
 &\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))(\Delta f(y) - \Delta f(x)) - \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 \\
 &\quad - \frac{1}{2}f(x) \sum_{y \in S_p^1} f(y) + \frac{1}{2}f(x)\Delta f(x) - \frac{1}{2}f^2(x) \\
 &= \frac{1}{2} \sum_{y \in V \sim x} (f(y) - f(x))(\Delta f(y) - \Delta f(x)) - \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 \\
 &\quad + \frac{1}{2}f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) - \frac{1}{2}f(x) \sum_{y \in S_p^1} f(y) + \frac{1}{2}f(x)\Delta f(x) - \frac{1}{2}f^2(x) \\
 &= \Gamma_1(f, \Delta f)(x) - \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 + \frac{1}{2}f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) \\
 &\quad - \frac{1}{2}f(x) \sum_{y \in S_p^1} f(y) + \frac{1}{2}f(x)\Delta f(x) - \frac{1}{2}f^2(x).
 \end{aligned}$$

(3) If $x = p$, then using (6)

$$\begin{aligned}
 \Gamma_1^c(f, \Delta^c f)(p) &= \frac{1}{2} \sum_{y \in C \sim p} (f(y) - f(p)) (\Delta^c f(y) - \Delta^c f(p)) \\
 &= \frac{1}{2} \sum_{y \in S_p^1} (f(y) - f(p)) (\Delta f(y) - f(y) - \sum_{z \in S_p^1} f(z)) \\
 &= \frac{1}{2} \sum_{y \in S_p^1} \left[f(y) \Delta f(y) - f^2(y) - f(y) \sum_{z \in S_p^1} f(z) \right] \\
 &= \frac{1}{2} \sum_{y \in S_p^1} f(y) \Delta f(y) - \frac{1}{2} \sum_{y \in S_p^1} f^2(y) - \frac{1}{2} \left(\sum_{y \in S_p^1} f(y) \right)^2.
 \end{aligned}$$

■

Lemma 3.6. *Let f be a function defined on the cone, and suppose $f(p) = 0$, then*

$$\Delta^c \Gamma_1^c(f)(x) = \begin{cases} \Delta \Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y); & x \in S_p^2 \\ \Delta \Gamma_1(f)(x) - \Gamma_1(f)(x) \\ \quad + \frac{1}{2} \left[\sum_{y \in S_p^1 \sim x} f^2(y) + \sum_{y \in S_p^1} f^2(y) \right] \\ \quad - \frac{1}{2} \deg(x) f^2(x) & x \sim p \\ \sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{|S_p^1| - 1}{2} \sum_{y \in S_p^1} f^2(y); & x = p \end{cases}$$

Proof. (1) If $x \in S_p^2$, then

$$\begin{aligned}
 \Delta^c \Gamma_1^c(f)(x) &= \sum_{y \in C \sim x} \left[\Gamma_1^c(f)(y) - \Gamma_1^c(f)(x) \right] \\
 &= \sum_{y \in V \setminus S_p^1 \sim x} \left[\Gamma_1^c(f)(y) - \Gamma_1^c(f)(x) \right] + \sum_{y \in S_p^1 \sim x} \left[\Gamma_1^c(f)(y) - \Gamma_1^c(f)(x) \right] \\
 &= \sum_{y \in V \setminus S_p^1 \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) \right] + \sum_{y \in S_p^1 \sim x} \left[\Gamma_1(f)(y) + \frac{1}{2} f^2(y) - \Gamma_1(f)(x) \right] \\
 &= \sum_{y \in V \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) \right] + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) \\
 &= \Delta \Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y).
 \end{aligned}$$

(2) If $x \sim p$, then

$$\begin{aligned}
 \Delta^c \Gamma_1^c(f)(x) &= \sum_{y \in C \sim x} \left[\Gamma_1^c(f)(y) - \Gamma_1^c(f)(x) \right] \\
 &= \sum_{y \in S_p^2 \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) - \frac{1}{2} f^2(x) \right] \\
 &\quad + \sum_{y \in S_p^1 \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) + \frac{1}{2} (f^2(y) - f^2(x)) \right] + \Gamma_1^c(f)(p) - \Gamma_1(f)(x) - \frac{1}{2} f^2(x) \\
 &= \sum_{y \in S_p^2 \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) \right] - \frac{1}{2} \deg(x) f^2(x) + \sum_{y \in S_p^1 \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) \right] \\
 &\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) + \frac{1}{2} \sum_{y \in S_p^1} f^2(y) - \Gamma_1(f)(x) \\
 &= \sum_{y \in V \sim x} \left[\Gamma_1(f)(y) - \Gamma_1(f)(x) \right] - \Gamma_1(f)(x) \\
 &\quad + \frac{1}{2} \sum_{y \in S_p^1 \sim x} f^2(y) + \frac{1}{2} \sum_{y \in S_p^1} f^2(y) - \frac{1}{2} \deg(x) f^2(x) \\
 &= \Delta \Gamma_1(f)(x) - \Gamma_1(f)(x) + \frac{1}{2} \left[\sum_{y \in S_p^1 \sim x} f^2(y) + \sum_{y \in S_p^1} f^2(y) \right] - \frac{1}{2} \deg(x) f^2(x).
 \end{aligned}$$

(3) If $x = p$, then

$$\begin{aligned}
 \Delta^c(\Gamma_1^c(f))(p) &= \sum_{y \in S_p^1} \left[\Gamma_1^c(f)(y) - \Gamma_1^c(f)(p) \right] \\
 &= \sum_{y \in S_p^1} \left[\Gamma_1(f)(y) + \frac{1}{2} f^2(y) - \frac{1}{2} \sum_{y \in S_p^1} f^2(y) \right] \\
 &= \sum_{y \in S_p^1} \Gamma_1(f)(y) + \frac{1}{2} \sum_{y \in S_p^1} f^2(y) - \frac{|S_p^1|}{2} \sum_{y \in S_p^1} f^2(y) \\
 &= \sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{(|S_p^1| - 1)}{2} \sum_{y \in S_p^1} f^2(y).
 \end{aligned}$$

■

Lemma 3.7. *Let f be a function defined on the cone, and suppose $f(p) = 0$, then*

$$\Gamma_2^c(f)(x) = \begin{cases} \Gamma_2(f)(x) + \frac{3}{4} \sum_{y \in S_p^1 \sim x} f^2(y) - \frac{1}{2} f(x) \sum_{y \in S_p^1 \sim x} f(y); & x \in S_p^2 \\ \Gamma_2(f)(x) - \frac{1}{2} \Gamma_1(f)(x) + \frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 \\ \quad + \frac{1}{4} \left[\sum_{y \in S_p^1 \sim x} f^2(y) - \deg(x) f^2(x) \right] - \frac{1}{2} f(x) \Delta f(x) \\ \quad - \frac{1}{2} f(x) \sum_{y \in S_p^2 \sim x} (f(y) - f(x)) + \frac{1}{4} \sum_{y \in S_p^1} f^2(y) + \frac{1}{2} f^2(x); & x \sim p \\ \frac{1}{2} \sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{1}{2} \sum_{y \in S_p^1} f(y) \Delta f(y) \\ \quad - \frac{|S_p^1| - 3}{4} \sum_{y \in S_p^1} f^2(y) + \frac{1}{2} \left(\sum_{y \in S_p^1} f(y) \right)^2; & x = p \end{cases}$$

Proof. (1) If $x \in S_p^2$, then

$$\begin{aligned}\Gamma_2^c(f)(x) &= \frac{1}{2}\Delta^c\Gamma_1^c(f, f)(x) - \Gamma_1^c(f, \Delta^c f)(x) \\ &= \frac{1}{2}\Delta\Gamma_1(f)(x) + \frac{1}{4}\sum_{y \in S_p^1 \sim x} f^2(y) - \Gamma_1(f, \Delta f)(x) + \frac{1}{2}\sum_{y \in S_p^1 \sim x} f(y)(f(y) - f(x)) \\ &= \Gamma_2(f)(x) + \frac{3}{4}\sum_{y \in S_p^1 \sim x} f^2(y) - \frac{1}{2}f(x)\sum_{y \in S_p^1 \sim x} f(y).\end{aligned}$$

(2) If $x \sim p$, then

$$\begin{aligned}\Gamma_2^c(f)(x) &= \frac{1}{2}\Delta^c\Gamma_1^c(f)(x) - \Gamma_1^c(f, \Delta^c f)(x) \\ &= \frac{1}{2}\Delta\Gamma_1(f)(x) - \frac{1}{2}\Gamma_1(f)(x) + \frac{1}{4}\sum_{y \in S_p^1 \sim x} f^2(y) + \frac{1}{4}\sum_{y \in S_p^1} f^2(y) - \frac{1}{4}\deg(x)f^2(x) \\ &\quad - \Gamma_1(f, \Delta f)(x) + \frac{1}{2}\sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 - \frac{1}{2}f(x)\sum_{y \in S_p^2 \sim x} (f(y) - f(x)) \\ &\quad + \frac{1}{2}f(x)\sum_{y \in S_p^1} f(y) - \frac{1}{2}f(x)\Delta f(x) + \frac{1}{2}f^2(x). \\ &= \left[\frac{1}{2}\Delta\Gamma_1(f)(x) - \Gamma_1(f, \Delta f)(x)\right] - \frac{1}{2}\Gamma_1(f)(x) + \frac{1}{2}\sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 \\ &\quad + \frac{1}{4}\left[\sum_{y \in S_p^1 \sim x} f^2(y) - \deg(x)f^2(x)\right] + \frac{1}{4}\sum_{y \in S_p^1} f^2(y) - \frac{1}{2}f(x)\sum_{y \in S_p^2 \sim x} (f(y) - f(x)) \\ &\quad - \frac{1}{2}f(x)\Delta f(x) + \frac{1}{2}f^2(x) \\ &= \Gamma_2(f)(x) - \frac{1}{2}\Gamma_1(f)(x) + \frac{1}{2}\sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 - \frac{1}{2}f(x)\sum_{y \in S_p^2 \sim x} (f(y) - f(x)) \\ &\quad + \frac{1}{4}\left[\sum_{y \in S_p^1 \sim x} f^2(y) - \deg(x)f^2(x)\right] - \frac{1}{2}f(x)\Delta f(x) + \frac{1}{4}\sum_{y \in S_p^1} f^2(y) + \frac{1}{2}f^2(x)\end{aligned}$$

(3) If $x = p$, then

$$\begin{aligned}\Gamma_2^c(f)(p) &= \frac{1}{2}\Delta^c\Gamma_1^c(f)(p) - \Gamma_1^c(f, \Delta^c f)(p) \\ &= \frac{1}{2}\sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{|S_p^1| - 1}{4}\sum_{y \in S_p^1} f^2(y) - \frac{1}{2}\sum_{y \in S_p^1} f(y)\Delta f(y) + \frac{1}{2}\sum_{y \in S_p^1} f^2(y) + \frac{1}{2}\left(\sum_{y \in S_p^1} f(y)\right)^2 \\ &= \frac{1}{2}\sum_{y \in S_p^1} \Gamma_1(f)(y) - \frac{1}{2}\sum_{y \in S_p^1} f(y)\Delta f(y) - \frac{|S_p^1| - 3}{4}\sum_{y \in S_p^1} f^2(y) + \frac{1}{2}\left(\sum_{y \in S_p^1} f(y)\right)^2.\end{aligned}$$

■

3.2. Γ_2^c for $C(G)$. When $C = (V^c, E^c)$ is the full cone over $V(G)$, then $S_p^1 = V$ and so Lemma 3.7 reduces to

Lemma 3.8.

$$\Gamma_2^c(f)(x) = \begin{cases} \Gamma_2(f)(x) + \Gamma_1(f)(x) + \frac{1}{4}\sum_{y \in V} f^2(y) + \frac{1}{2}f^2(x); & x \sim p \\ \sum_{y \in V} \Gamma_1(f)(y) - \frac{|V| - 3}{4}\sum_{y \in V} f^2(y) + \frac{1}{2}\left(\sum_{y \in V} f(y)\right)^2; & x = p \end{cases}.$$

Proof. Since $S_p^1 = V$ and $S_p^2 = \emptyset$ the first case in Lemma 3.7 disappears. In case 2 notice that when $S_p^1 = V$, then $\frac{1}{2} \sum_{y \in S_p^1 \sim x} (f(y) - f(x))^2 = \Gamma_1(f)(x)$ and $\sum_{y \in S_p^1 \sim x} (f^2(y) - f^2(x)) = (\Delta f^2)(x)$. Since $\Gamma_1(f)(x) = \frac{1}{2} [(\Delta f^2)(x) - f(x) \Delta f(x)]$ the case when $x \sim p$ follows. When $x = p$, applying the identity (7) gives the desired result. ■

This leads to the following result regarding the curvature of the cone,

Proof of Theorem 1.6. Suppose G satisfies $CD(K, \infty)$ for $K \leq \frac{1}{2}$. Since G satisfies $CD(K, \infty)$ then by lemma 3.8 for $x \sim p$,

$$\Gamma_2^c(f)(x) \geq (K+1)\Gamma_1(f)(x) + \frac{1}{4} \sum_{y \in V} f^2(y) + \frac{1}{2} f^2(x).$$

Therefore,

$$\Gamma_2^c(f)(x) \geq (K+1)\Gamma_1^c(f)(x) + \frac{1}{4} \sum_{y \in V} f^2(y) - \frac{K}{2} f^2(x).$$

Since $K \leq \frac{1}{2}$, then $\frac{1}{4} \sum_{y \in V} f^2(y) - \frac{K}{2} f^2(x) \geq 0$. Hence we may drop both terms from the inequality and $C(G)$ satisfies $CD(K+1, \infty)$. ■

4. $CCD(K, N)$ AND GLOBAL POINCARÉ INEQUALITY

If the cone C satisfies the $CD(K, N)$ inequality at the vertex, p , then by Lemmas 3.3 and 3.2, we get

$$\Gamma_2^c(f)(p) \geq \frac{1}{N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{K}{2} \sum_{y \in V} f^2(y). \quad (10)$$

Proof of Theorem 1.1. Suppose a graph G satisfies $CCD(K, N)$. By Lemma 3.8,

$$\Gamma_2^c(f)(p) = \sum_{y \in V} \Gamma_1(f)(y) - \frac{|V|-3}{4} \sum_{y \in V} f^2(y) + \frac{1}{2} \left(\sum_{y \in V} f(y) \right)^2, \quad (11)$$

Upon combining (11) and (10), we will arrive at

$$\sum_{y \in V} \Gamma_1(f)(y) - \frac{|V|-3}{4} \sum_{y \in V} f^2(y) + \frac{1}{2} \left(\sum_{y \in V} f(y) \right)^2 \geq \frac{1}{N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{K}{2} \sum_{y \in V} f^2(y),$$

which simplifies to

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K+|V|-3}{4} \sum_{y \in V} f^2(y).$$

For f with $\text{avg}(f) = 0$, the above reduces to

$$\frac{1}{2} \sum_{y \in V} |\nabla f(y)|^2 \geq \frac{2K+|V|-3}{4} \sum_{y \in V} f^2(y).$$

By the definition in 6 this yields the Poincaré inequality,

$$\|f\|_2 \leq \sqrt{\frac{2}{2K+|V|-3}} \|\nabla f\|_2, \text{ when } \text{avg}(f) = 0. \quad \blacksquare$$

Proof of Theorem 1.3. Suppose a finite graph G satisfies the $CCD(K, N)$ and f is a non-zero harmonic function, then one has $\sum_{y \in V} \Gamma_1(f)(y) = 0$ (i.e. f is constant on connected components). Thus,

$$\frac{2K + |V| - 3}{4} \sum_{y \in V} f^2(y) \leq \frac{N - 2}{2N} \left(\sum_{y \in V} f(y) \right)^2.$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{y \in V} f(y) \right)^2 \leq |V| \cdot \sum_{y \in V} f^2(y),$$

which implies

$$\frac{2K + |V| - 3}{4} \sum_{y \in V} f^2(y) \leq \frac{N - 2}{2N} |V| \cdot \sum_{y \in V} f^2(y).$$

Since, f is not constant zero,

$$K \leq \frac{|V|}{2} - \frac{2|V|}{N} + \frac{3}{2}.$$

■

Having established an upper bound for the curvature at the cone point over the vertex set of the graph G we now turn to an investigation of when the maximum curvature value is achieved.

Lemma 4.1. *For any finite graph, G , the Ricci curvatures $\text{Ric}_\infty(G)$, $\text{Ric}_N(G)$, $CRic_\infty(G)$ and $CRic_N(G)$ are realized by some functions, i.e. there are functions that achieve the equality in the (corresponding) defining Bakry-Émery curvature-dimension inequalities.*

Proof. We will just present a proof for $\text{Ric}_N(G)$. The proofs for other Ricci curvatures are similar.

Since, $\text{Ric}_N(G)$ is the supremum of all possible lower curvature bounds, one can find a sequence, g_i such that for all $v \in V$

$$\frac{1}{N} (\Delta g_i)^2(v) + \text{Ric}_N(G) \Gamma_1(g_i)(v) \leq \Gamma_2(g_i)(v) < \frac{1}{N} (\Delta g_i)^2(v) + \left(\text{Ric}_N(G) + \frac{1}{i} \right) \Gamma_1(g_i)(v). \quad (12)$$

All the terms appearing in the above inequality are invariant under rescaling of the g_i 's. Hence, without loss of generality, we may assume that $\text{Range}(g_i) \subset [-1, 1]$, for all i . Now since $V(G)$ is finite then by a diagonal argument one can find a subsequence g_j of the g_i 's that converge to a function g . Taking the limit of (12) as $j \rightarrow \infty$ shows that g achieves $\text{Ric}_N(G)$. ■

5. FUNCTIONS THAT MAXIMIZE THE CONICAL CURVATURE

In this section we will show that functions that maximize the conical curvature are generalized harmonic (see Definition 2.5) when the underlying graph is k -regular for k sufficiently large (compared to $|V|$).

Proof of Theorem 1.5. Suppose G satisfies $CCD(K_{max}^c, N)$, then for any f

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2 - N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K + |V| - 3}{4} \sum_{y \in V} f^2(y).$$

Since $K_{max}^c = \frac{|V|}{2} + \frac{3}{2} - 2\frac{|V|}{N} = \frac{N \cdot |V| + 3N - 4|V|}{2N}$, this simplifies to

$$\begin{aligned} \sum_{y \in V} \Gamma_1(f)(y) &\geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{N \cdot |V| + 3N - 4|V| + N \cdot |V| - 3N}{4N} \sum_{y \in V} f^2(y) \\ &\geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{N \cdot |V| - 2|V|}{2N} \sum_{y \in V} f^2(y) \\ &\geq \frac{2-N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{N-2}{2N} \cdot |V| \sum_{y \in V} f^2(y) \\ &\geq \frac{N-2}{2N} \left[|V| \sum_{y \in V} f^2(y) - \left(\sum_{y \in V} f(y) \right)^2 \right]. \end{aligned}$$

Take $\varphi : V \rightarrow \mathbb{R}$ to be any variation function on the vertex set of G and let $t \in \mathbb{R}$, then

$$\sum_{y \in V} \Gamma_1(f + t\varphi)(y) \geq \frac{N-2}{2N} \left[|V| \sum_{y \in V} (f + t\varphi)^2(y) - \left(\sum_{y \in V} (f + t\varphi)(y) \right)^2 \right].$$

For an f that achieves K_{max}^c , one has $\frac{d}{dt}|_{t=0}$ of both sides are equal for any variation φ . Hence a straightforward calculation yields the linearized equation,

$$\sum_{y \in V} \sum_{z \sim y} (f(z) - f(y))(\varphi(z) - \varphi(y)) = \frac{N-2}{2N} \left[|V| \sum_{y \in V} f(y)\varphi(y) - \sum_{y \in V} f(y) \sum_{y \in V} \varphi(y) \right]. \quad (13)$$

Now for fixed $r \in V$ and $\varphi(y) = \delta_r(y)$, we get

$$\sum_{y \in V} \sum_{z \sim y} (f(z) - f(y))(\delta_r(z) - \delta_r(y)) = -2\Delta f(r).$$

Thefore, (13) reduces to

$$-2\Delta f(r) = \frac{N-2}{2N} \left[|V|f(r) - \sum_{y \in V} f(y) \right],$$

which is equivalent to

$$\Delta f(r) = \frac{N-2}{4N} \bar{\Delta} f(r), \quad (14)$$

where $\bar{\Delta}$ denotes the Laplacian for the graph completion, \bar{G} , of G . When G is a complete graph, this right away implies that $\Delta f(r) = 0$.

In general, the equation (14) gives

$$\Delta(f - \text{avg}(f))(r) = -\frac{N-2}{4N} |V| (f - \text{avg}(f))(r).$$

Now if f is not constant then by the Rayleigh quotient, (9), we see that $\lambda_1(G) = \frac{N-2}{4N} |V|$ and $f - \text{avg}(f)$ is an eigenfunction for λ_1 .

For the "if" direction suppose for some non-constant function, f , that $f - \text{avg}(f)$ is an eigenfunction for $\lambda_1 = \frac{N-2}{4N} |V|$. Tracing back the above computations one has (13) holds for $\varphi = \delta_y$'s. Then since (13) is linear in φ , one can use $f = \sum_{y \in V} f(y)\delta_y$ instead of φ which will translate to f realizing K_{max}^c . ■

The last assertion in Theorem 5 is due to the following general lemma

Lemma 5.1. *Let G be a k -regular graph and f an eigenfunction corresponding to the eigenvalue λ_1 such that $\text{avg}(f) = 0$. If $\lambda_1 < 2k$, then f is generalized harmonic of both types I and II.*

Proof. By our hypothesis, $\Delta f = -\lambda_1 f$, hence for any $r \in V$,

$$\sum_{z \sim r} f(z) - \deg(r)f(r) = -\lambda_1 f(r),$$

and so,

$$(k - \lambda_1)f(r) = \sum_{z \sim r} f(z) = kg_1(r).$$

Therefore, g_1 is again an eigenfunction corresponding to λ_1 and the same argument can be applied, yielding

$$(k - \lambda_1)^2 f(r) = k(k - \lambda_1)g_1(r) = k^2 g_2(r).$$

Iterating this construction, we will get

$$(k - \lambda_1)^n f(r) = k^n g_n(r),$$

or

$$g_n(r) = \left(\frac{k - \lambda_1}{k} \right)^n f(r).$$

Since we assumed $0 < \lambda_1 < 2k$ then $\left| \frac{k - \lambda_1}{k} \right| < 1$. Taking the limit as $n \rightarrow \infty$ the result follows. To prove that f is generalized harmonic of type II we notice that

$$(k + 1 - \lambda_1)f(r) = (k + 1)h_1(r).$$

A similar iteration argument as above will lead to

$$h_n(r) = \left(\frac{\lambda_1 + 1}{k + 1} \right)^n f(r),$$

which is again converging to 0 (= avg(f) after normalization). ■

Remark 5.2. The condition that $0 < \lambda_1 < 2k$ holds whenever G is connected and not bipartite (see [5, Proposition 0.5]).

6. $CCD(K, N)$ AND LOWER BOUNDS ON $\lambda_1(G)$

In this section we assume that a given graph, G , satisfies the $CCD(K, N)$ condition. We will use the resulting global Poincaré inequality along with the results of the last section to find lower bounds on the first non-zero eigenvalues of such graphs.

Lemma 6.1. Suppose G satisfies $CCD(K, N)$ and $N \geq 2$. Then Cheeger's isoperimetric constant, $h(G)$, satisfies

$$h(G) \geq \frac{|V| + 2NK - 3N}{4N},$$

and consequently,

$$\lambda_1(G) \geq \frac{(|V| + 2NK - 3N)^2}{32N^2 d_{max}}.$$

Proof. Since G satisfies the $CCD(K, N)$, for any f , we have the global Poincaré inequality (see Theorem 1.1)

$$\sum_{y \in V} \Gamma_1(f)(y) \geq \frac{2 - N}{2N} \left(\sum_{y \in V} f(y) \right)^2 + \frac{2K + |V| - 3}{2} \sum_{y \in V} f^2(y). \quad (15)$$

Suppose $F \subset V$ and $|F| \leq \frac{|V|}{2}$. Let $f = \chi_F$ be the characteristic function of F , then (15) becomes

$$\frac{2 - N}{2N} |F|^2 + \frac{2K + |V| - 3}{2} |F| \leq 2|\partial F|.$$

Hence, when $N \geq 2$,

$$\frac{|\partial F|}{|F|} \geq \frac{2-N}{4N} |F| + \frac{2K+|V|-3}{4} \geq \frac{2-N}{4N} \frac{|V|}{2} + \frac{2K+|V|-3}{4} = \frac{|V|+2NK-3N}{4N}.$$

Now applying Cheeger's inequality (see [1] and [2]), we know that $\lambda_1(G) \geq \frac{h^2(G)}{2d_{\max}}$, where d_{\max} is the maximum degree in the graph, G . Hence,

$$\lambda_1 \geq \frac{h^2(G)}{2d_{\max}} \geq \frac{(|V|+2NK-3N)^2}{32N^2d_{\max}}.$$

■

In the rest of this section we show that any lower bound, λ , for $\lambda_1(G)$ will imply that G satisfies $CCD(K, N)$ for some K and N (depending on λ).

Theorem 6.1. *Suppose $\lambda_1(G) \geq \lambda$, then G satisfies $CCD(K, N)$ for any K and N with*

$$N \geq 2 + \frac{2\lambda}{|V|-\lambda}, \quad \text{and} \quad K \leq \lambda - \left(\frac{|V|-3}{2}\right).$$

Proof. Since $\lambda_1(G) \geq \lambda$ then by the Rayleigh quotient, (9), we get

$$\begin{aligned} \sum_{y \in V} \Gamma_1(f)(y) &\geq \frac{\lambda}{2} \sum_{y \in V} (f - \text{avg}(f))^2(y) = \frac{\lambda}{2} \left[\sum_{y \in V} f^2(y) + |V| \text{avg}(f)^2 - 2 \text{avg}(f) \sum_{y \in V} f(y) \right] \\ &= \frac{\lambda}{2} \sum_{y \in V} f^2(y) + \frac{\lambda}{2|V|} \left(\sum_{y \in V} f(y) \right)^2 - \frac{\lambda}{|V|} \left(\sum_{y \in V} f(y) \right)^2 \\ &= \frac{\lambda}{2} \sum_{y \in V} f^2(y) - \frac{\lambda}{2|V|} \left(\sum_{y \in V} f(y) \right)^2. \end{aligned} \tag{16}$$

Comparing (16) to the global Poincaré inequality, (15) caused by $CCD(K, N)$, and one observes that G satisfies $CCD(K, N)$ for any K, N with

$$\frac{\lambda}{|V|} \leq \frac{N-2}{N}, \quad \text{and} \quad K \leq \lambda - \left(\frac{|V|-3}{2}\right).$$

The conclusion follows by noticing that one always have $\lambda_1(G) \leq |V|$.

■

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